Yiddish word of the day

"oysgemutchet" = אויסגעמוטשעט

exhausted =

Yiddish expression of the day

"Zol lign in der erd un bakn bagel" = זאָל ליגן אין דער
ערד און באַקן בייגל!

may you lie in the ground and bake bagels. =

Matrix Multiplication

1st - Matrix Addition

Let A, B be $m \times n$ matrices.

$$A = (a_{ij})_{\substack{i=1,n \\ j=1,m}} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, \quad B = (b_{ij})$$

Then

$$A + B = (a_{ij} + b_{ij}) = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ \vdots & & & \\ a_{m1}+b_{m1} & & \cdots & a_{mn}+b_{mn} \end{pmatrix}_{m \times n}$$

note: If we think of vector $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ in $\mathbb{R}^n$

we can think of this as $n \times 1$ matrix

So this addition we defined for matrices just generalizes the addition we defined for vectors.

ex) $$\underset{2\times 3}{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{pmatrix}} + \underset{2\times 3}{\begin{pmatrix} 0 & 1 & 2 \\ 6 & 8 & 1 \end{pmatrix}} = \underset{2\times 3}{\begin{pmatrix} 1 & 3 & 5 \\ 8 & 12 & 2 \end{pmatrix}}$$

There's another reason to define matrix add this way

Recall: $T: \mathbb{R}^n \to \mathbb{R}^m \xrightarrow{\text{we get}} A_T$ ($m \times n$ matrix)

Now, let $T, S: \mathbb{R}^n \to \mathbb{R}^m$ be two linear transf.
we get a new transformation

$$(T+S): \mathbb{R}^n \to \mathbb{R}^m \quad \text{defined by}$$

$$(T+S)(\vec{v}) = T(\vec{v}) + S(\vec{v})$$

$T \rightsquigarrow A_T$ $\qquad (T+S) \rightsquigarrow A_{T+S}$

$S \rightsquigarrow A_S$

Then the way we've defined matrix addition makes

$$\boxed{A_{T+S} = A_T + A_S}$$

ex) $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ y \end{pmatrix}$

$S : \mathbb{R}^2 \to \mathbb{R}^3$ by $S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x \\ y \end{pmatrix}$

Then $(T+S)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ y \end{pmatrix} + \begin{pmatrix} x \\ x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ x+y \\ 2y \end{pmatrix}$

Find A_{T+S}, A_S, A_T check $A_{T+S} = A_S + A_T$

$$A_T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_S = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underbrace{A_{T+S} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 2 \end{pmatrix}}_{A_{T+S}} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}}_{A_T} + \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}}_{A_S}$$

ex) $T : \mathbb{R}^3 \to \mathbb{R}^5 \qquad T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ x_2 \\ 0 \\ x_3 \end{pmatrix}$

$S : \mathbb{R}^3 \to \mathbb{R}^5 \qquad S\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 \\ 2x_2 \\ 0 \\ x_3 \end{pmatrix}$

$$(T+S) : \mathbb{R}^3 \to \mathbb{R}^5, \quad (T+S)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} x_1 \\ 0 \\ x_2 \\ 0 \\ x_3 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_1 \\ 2x_2 \\ 0 \\ x_3 \end{pmatrix}$$

$$\boxed{(T+S)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ x_1 \\ 3x_2 \\ 0 \\ 2x_3 \end{pmatrix}}$$

$$A_T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\underline{5} \, \underline{\times} \, \underline{3}} \qquad A_S = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{5 \times 3}$$

$$A_T + A_S = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \overset{\text{claim}}{=} A_{T+S}$$

$$A_{TS} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \underline{A_T} + \underline{A_S} \quad \underline{\checkmark}$$

Recall - Function Composition

$f: A \to B$, $g: B \to C$ we get a new function

$g \circ f : A \to C$ defined by $(g \circ f)(a) = g(f(a))$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

ex) $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$

$g: \mathbb{R} \to \mathbb{R}$ by $g(x) = x - 2$

Then $g \circ f : \mathbb{R} \to \mathbb{R}$, $(g \circ f)(x) = g(x^2) = x^2 - 2$

ii) $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ by $f(x) = \begin{pmatrix} 2x \\ x^2 \end{pmatrix}$

$g : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ by $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$

$$\begin{tikzcd} \mathbb{R} \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & \mathbb{R}^2 \arrow[r, "g"] & \mathbb{R}^3 \end{tikzcd}$$

$g \circ f : \mathbb{R} \longrightarrow \mathbb{R}^3$

$$(g \circ f)(x) = g\begin{pmatrix} 2x \\ x^2 \end{pmatrix} = \begin{pmatrix} 2x \\ x^2 \\ 0 \end{pmatrix}$$

$$(g \circ f)(3) = g\begin{pmatrix} 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ 0 \end{pmatrix}$$

Note: In example 2 can't do $f \circ g$

In example 1, we can do $f \circ g$ as well

but $f \circ g \neq g \circ f$

- $(f \circ g)(x) = f(x-2) = (x-2)^2 = x^2 - 4x + 4$

$\neq x^2 - 2$

$\underline{\text{Now}}$: Suppose $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m \xrightarrow{S} \mathbb{R}^q$, $\underbrace{\qquad\qquad\qquad}_{S \circ T}$

that $T: \mathbb{R}^n \to \mathbb{R}^m$, $S: \mathbb{R}^m \to \mathbb{R}^q$ are linear transf.

Then the function $S \circ T : \mathbb{R}^n \to \mathbb{R}^q$ is still a linear transformation (you can check this)

Really! Check this!!

$T: \mathbb{R}^n \to \mathbb{R}^m \rightsquigarrow \underline{A_T}$ ($\underline{m \times n}$ matrix)

$S: \mathbb{R}^m \to \mathbb{R}^q \rightsquigarrow \underline{A_S}$ ($\underline{q \times m}$ matrix)

$S \circ T: \mathbb{R}^n \to \mathbb{R}^q \rightsquigarrow \underline{A_{S \circ T}}$ ($\underline{q \times n}$ matrix)

Q: How are these 3 matrices related?

- Recall that given $T: \mathbb{R}^n \to \mathbb{R}^m$ we have the corresponding matrix A_T and the property that $T(\vec{v}) = A_T\vec{v}$
- So the matrix associated to $S \circ T$ should satisfy a similar property as above
  - That is, we want $(S \circ T)(\vec{v}) = A_{S \circ T}\vec{v}$

We will define matrix mult in such a way, that this relationship holds

Let A $m \times n$ matrix B $n \times q$ matrix

$\downarrow$ $\downarrow$

$T: \mathbb{R}^n \to \mathbb{R}^m$ $S: \mathbb{R}^q \to \mathbb{R}^n$

AB will be a $m \times q$ matrix whose $(i,j)^{th}$ component will be

- "multiply" row i of matrix A to "column j" of matrix B

Examples

$A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}_{2\times 2}$ $B = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}_{2\times 2}$

$\Rightarrow$ AB should be a $\underline{2\times 2}$ matrix

$(1,1)$: $\underline{1\cdot 2 + 0\cdot 1}$

$(1,2)$: $\underline{1\cdot 3 + 0\cdot 0}$

$(2,1)$: $\underline{1\cdot 2 + 2\cdot 1}$

$(2,2)$: $\underline{1\cdot 3 + 2\cdot 0}$

$\Longrightarrow \boxed{AB = \begin{pmatrix} 2 & 3 \\ 4 & 3 \end{pmatrix}}$

$B = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}_{2\times 2}, \quad A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}_{2\times 2}$ BA 2x2 matrix

$(1,1): \underline{2\cdot 1 + 3\cdot 1}$

$(1,2): \underline{2\cdot 0 + 3\cdot 2}$

$(2,1): \underline{1\cdot 1 + 0\cdot 1}$

$(2,2): \underline{1\cdot 0 + 0\cdot 2}$

$\Rightarrow BA = \begin{pmatrix} 5 & 6 \\ 1 & 0 \end{pmatrix}$

$$\underline{AB \neq BA}$$

Check: That this def of multiplication gives us what we want

ex) $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \end{pmatrix}$

$S: \mathbb{R}^2 \to \mathbb{R}^2$ by $S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$

Then $T \circ S: \mathbb{R}^2 \to \mathbb{R}^2$ by $(T \circ S)\begin{pmatrix} x \\ y \end{pmatrix} = T\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} y+x \\ x \end{pmatrix}$

That is $(T \circ S)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y+x \\ x \end{pmatrix}$

$T \longrightarrow A_T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}_{2\times 2}$

$S \longrightarrow A_S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{2\times 2}$

$$T \circ S \longrightarrow A_{T\circ S} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}_{2\times 2}$$

Claim is : $A_{T\circ S} = A_T A_S$

$$\underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{A_T} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{A_S} = \begin{pmatrix} 1\cdot 0 + 1\cdot 1 & 1\cdot 1 + 1\cdot 0 \\ 0\cdot 0 + 1\cdot 1 & 0\cdot 1 + 1\cdot 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= A_{T\circ S} \quad \checkmark$$

Reverse way

$$(S\circ T):\mathbb{R}^2\to\mathbb{R}^2 \quad \text{by} \quad (S\circ T)\begin{pmatrix}x\\y\end{pmatrix} = S\begin{pmatrix}x+y\\y\end{pmatrix} = \begin{pmatrix}y\\x+y\end{pmatrix}$$

$$S\circ T \longrightarrow A_{S\circ T} = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$$

Now check that $A_{S\circ T} = A_S A_T$

$$\underbrace{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}}_{A_S}\underbrace{\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}}_{A_T} = \begin{pmatrix}0\cdot1+1\cdot0 & 0\cdot1+1\cdot1\\ 1\cdot1+0\cdot0 & 1\cdot1+0\cdot1\end{pmatrix}$$

$$= \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix} \checkmark$$

Ways matrix mult is different than "regular" multiplication

1) we've seen it's not commutative, ie $AB \neq BA$

for many reasons!

ex) $A = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix}_{2\times 3}$ $B = \begin{pmatrix} 1 & 6 \\ 2 & 0 \\ 3 & 0 \end{pmatrix}_{3\times 2}$

$$AB = \begin{pmatrix} 1\cdot 1 + 1\cdot 2 + 2\cdot 3 & 1\cdot 6 + 0 \\ 3\cdot 1 + 0\cdot 2 + 1\cdot 3 & 3\cdot 6 + 0 \end{pmatrix} = \begin{pmatrix} 9 & 6 \\ 6 & 18 \end{pmatrix}_{2\times 2}$$

$$BA = \begin{pmatrix} 1\cdot 1 + 6\cdot 3 & 1\cdot 1 + 6\cdot 0 & 1\cdot 2 + 6\cdot 1 \\ 2\cdot 1 + 0\cdot 3 & 2\cdot 1 + 0\cdot 0 & 2\cdot 2 + 0\cdot 1 \\ 3\cdot 1 + 0\cdot 3 & 3\cdot 1 + 0\cdot 0 & 3\cdot 2 + 0\cdot 1 \end{pmatrix} = \begin{pmatrix} 19 & 1 & 8 \\ 2 & 2 & 4 \\ 3 & 3 & 6 \end{pmatrix}_{3\times 3}$$

ex2: $A = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix}_{2\times 3}$ $\qquad B = \begin{pmatrix} 1 & 6 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 1 \end{pmatrix}_{3\times 3}$

Note: Cannot do BA, not defined!

$$AB = \begin{pmatrix} 1\cdot 1 + 1\cdot 2 + 2\cdot 3 & 1\cdot 6 + 0 & 1\cdot 0 + 1\cdot 0 + 2\cdot 1 \\ 3\cdot 1 + 0\cdot 2 + 1\cdot 3 & 3\cdot 6 + 0 & 3\cdot 0 + 0\cdot 0 + 1\cdot 1 \end{pmatrix} = \begin{pmatrix} 9 & 6 & 2 \\ 6 & 18 & 1 \end{pmatrix}_{2\times 3}$$

2) It's possible for 2 non-zero matrices multiply to 0 matrix

(shows that matrices aren't a so called integral domain) ex) $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_{2\times 2}$$

3) Can't "cancel" out matrices.

ex) $A = \begin{pmatrix} 2 & -3 \\ -4 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $C = \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}$

$AB = \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix} \Rightarrow AB = AC$ but $B \neq C$

$AC = \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}$